Homework 2 Solutions

Due: Thursday September 13th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Let E, F be measurable sets in $\mathbb{R}$. Show

$$m(E \cup F) = m(E) + m(F) - m(E \cap F).$$

Solution:

$$m(E \cup F) = m(E - F \cap E) + m(F - E \cap F) + m(E \cap F) =$$

$$m(E) - m(E \cap F) + m(F) - m(E \cap F) - m(E \cap F) = m(E) + m(F) - m(E \cap F).$$

Problem 2: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Show that $f^{-1}(I)$ is measurable for any interval I.

Solution: The interval I is a countable intersection of open intervals $\cap_{n \in \mathbb{N}} I_n$. Now $f^{-1}(I_n)$ is open and hence measurable and hence

$$f^{-1}(I) = f^{-1}(\cap_{n \in \mathbb{N}} I_n) = \cap_{n \in \mathbb{N}} f^{-1}(I_n)$$

is measurable since it is a countable intersection of measurable sets.

Problem 3: Show that $E \subset \mathbb{R}$ is measurable iff

$$m^*(I) = m^*(E \cap I) + m^*(I - E \cap I)$$

for every interval I.

Solution: If E is measurable then

$$m^*(I) = m^*(E \cap I) + m^*(I - E \cap I)$$

holds by definition.

Now suppose that

$$m^*(I) = m^*(E \cap I) + m^*(I - E \cap I)$$

for every interval I. We wish to show that E is measurable. Let $A \subset \mathbb{R}$ be any subset. It is sufficient to show that

$$m^*(E \cap A) + m^*(A - E \cap A) \leq m^*(A) + \epsilon \tag{1}$$

for each $\epsilon > 0$ since m^* is subadditive. Therefore, fix $\epsilon > 0$. Choose an interval covering $(I_n)_{n \in \mathbb{N}}$ of A so that $\sum_{n \in \mathbb{N}} l(I_n) < m^*(A) + \epsilon$. Now

$$m^*(A) + \epsilon > \sum_{n \in \mathbb{N}} l(I_n) = \sum_{n \in \mathbb{N}} m^*(I_n) = \sum_{n \in \mathbb{N}} (m^*(E \cap I_n) + m^*(I_n - (E \cap I_n))) =$$

$$\sum_{n \in \mathbb{N}} m^*(E \cap I_n) + \sum_{n \in \mathbb{N}} m^*(I_n - (E \cap I_n)) \geq$$

$$
\begin{aligned}
m^*(\cup_{n\in\mathbb{N}}(E\cap I_n)) + m^*(\cup_{n\in\mathbb{N}}(I_n - (E\cap I_n))) \\
\geq m^*(E\cap(\cup_{n\in\mathbb{N}} I_n)) + m^*(\cup_{n\in\mathbb{N}} I_n - (E\cap(\cup_{n\in\mathbb{N}} I_n))) \\
\geq m^*(E\cap A) + m^*(A - E\cap A).
\end{aligned}
$$

Hence Equation (1) holds and we are done.

Problem 4: **(Optional)** Let $A \subset \mathbb{R}$ be a subset satisfying the following two properties:

- $a - b \notin \mathbb{Q} - \{0\}$ for all $a, b \in A$ and
- for each $x \in \mathbb{R}$ there exists $a \in A$ so that $x - a \in \mathbb{Q}$.

Such a set exists by the axiom of choice. Show that A is not measurable.

Solution: We define

$$B + t := \{b + t \ : \ b \in B\}, \quad B - t := \{b - t \ : \ b \in B\}$$

for each subset $B \subset \mathbb{R}$ and $t \in \mathbb{R}$.

Suppose (for a contradiction) that A is measurable. Then for each $q \in (0,1) \cap \mathbb{Q}$

$$A_q := \cup_{n\in\mathbb{Z}}((A \cap [nq, (n+1)q)) - nq) \subset [0, q)$$

is measurable. Also since the sets $(A \cap [nq, (n+1)q)) - nq$, $n \in \mathbb{Z}$ are pairwise disjoint since $nq \in \mathbb{Q}$ and also $(A \cap [nq, (n+1)q))$, $n \in \mathbb{Z}$ are pairwise disjoint, we have that

$$m(A) = \sum_{n\in\mathbb{Z}} (A \cap [nq, (n+1)q)) = \sum_{n\in\mathbb{Z}} ((A \cap [nq, (n+1)q)) - nq) = m(A_q).$$

Hence $m(A) = m(A_q) \leq m([0, q)) = q$. Hence $m(A) < q$ for each rational $q \in (0, 1)$. Therefore $m(A) = 0$. However since the sets $(A + q)_{q\in\mathbb{Q}}$ are pairwise disjoint, we have

$$\infty = m(\mathbb{R}) = m(\cup_{q\in\mathbb{Q}}(A + q)) = \sum_{q\in\mathbb{Q}} m(A + q) = \sum_{q\in\mathbb{Q}} m(A) = 0.$$

Contradiction.